Approximation Algorithms

Lecture 9

Last Time:

Uncapacitated Facility Location

Today:

Bin packing

<u>Bin Packing</u>

- n items ; $a_1, \dots, a_n \in (0,1]$ denote their sizes
- Pack items into as few bins of size 1 as possible

[Karmarkar & Karp '78]

Today : $OPT + O(\log^2 OPT)$ approx. algo.

based on deterministic rounding

Assumption : Each piece has size $\geq \frac{1}{SIZE(I)}$

- Let $s_1, \dots, s_m$ be the distinct item sizes in input I.

$$s_m \geq \frac{1}{\text{SIZE}(I)}$$

- $b_i \triangleq$ # items of size s_i for $i \in [m]$

- Configuration: m-tuple $(t_1, \dots, t_m)$ denoting packing of a bin

  i.e, this bin has t_i items of size s_i $\forall i \in [m]$

  $$\sum t_i s_i \leq 1$$

- # possible configurations ?
- Let $T_1, \dots, T_N$ denote the configurations.
- variable x_j for T_j, $j \in [N]$
  denotes # bins packed with config T_j.
- $t_{ij} \triangleq$ #items of size s_i in config T_j
  $i \in [m]$, $j \in [N]$

IP formulation for bin packing

T_j $\quad$ t_{ij}

items of size s_i in T_j

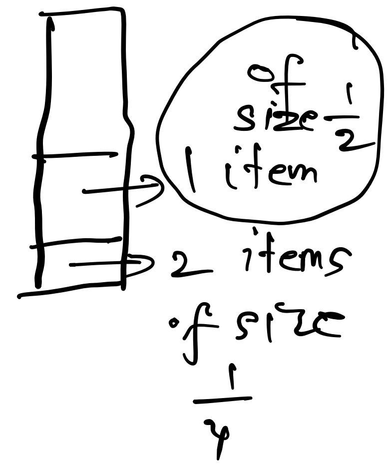
minimize $\sum_{j=1}^{N} x_j$

s.t.

$$\sum t_{ij} x_j \geq b_i \quad \forall i \in [m]$$

$$x_j \in \mathbb{N} \cup \{0\}$$

T_j



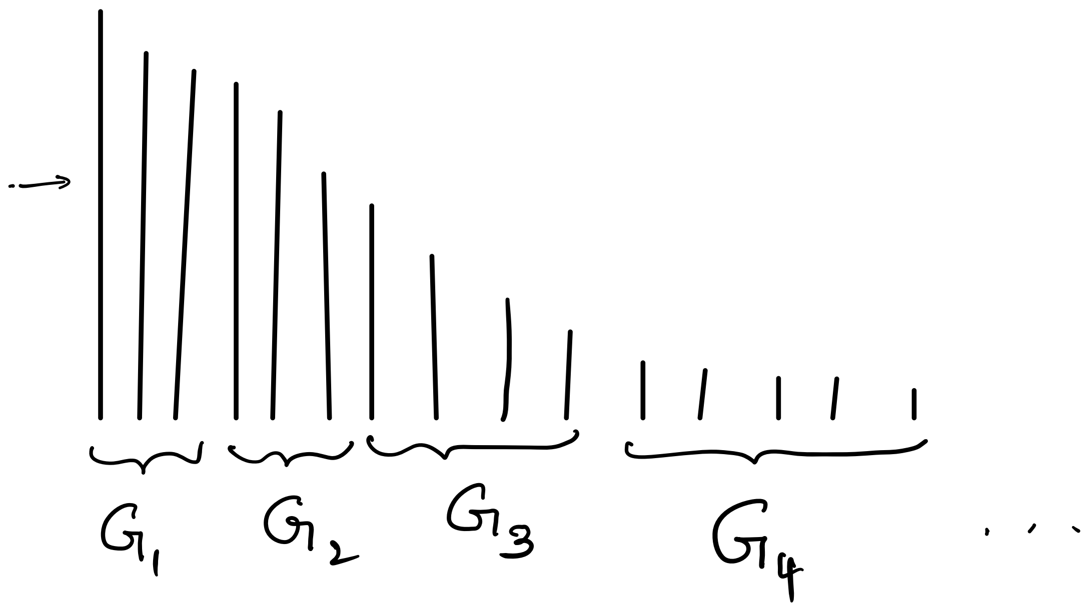
- Clearly, $\sum s_i b_i = SIZE(I) \leq \underbrace{Z^*_{LP}(I)} \leq OPT(I)$

bin packing instance

Theorem (Karmarkar & Karp)

LP can be solved up to additive error of 1 in time $\mathrm{poly}\left(m, \log\left(\frac{n}{s_m}\right)\right)$

Next ingredient: Harmonic grouping scheme

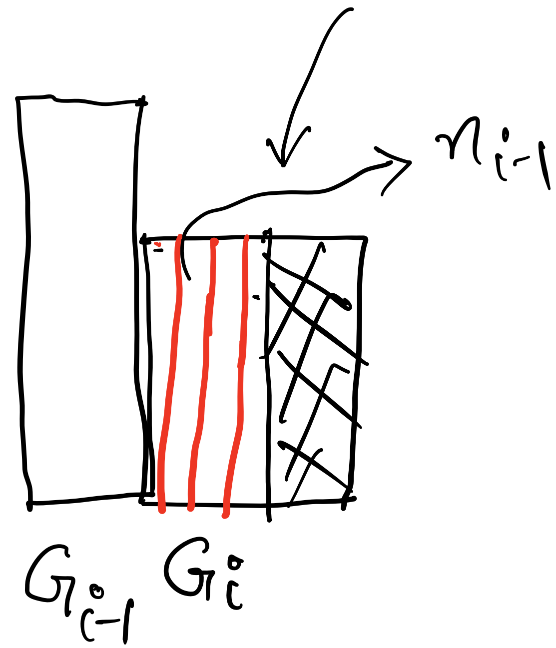


r groups

- G_1 consists of the smallest set of largest items whose sizes sum up to ≥ 2
- . . . . . .

$$\Rightarrow \; r \leq \frac{SIZE(I)}{2}$$

n_i - # items in G_i



- $n_i \geq n_{i-1} \quad \forall i = 2, \ldots, r-1$
- Discard G_1 & G_r
  $\equiv$ retain largest n_{i-1} items in G_i
- for each $i \in \{2, \ldots, r-1\}$
  - discard the smallest $n_i - n_{i-1}$ items in G_i
  - equalize sizes of remaining items to the largest size in G_i

Rounded instance − I'

Claim A: # distinct item sizes in $I' \leq \frac{SIZE(I)}{2}$

Claim B: Total size of all discarded pieces is $O(\log(SIZE(I)))$

Pf: Consider items removed from G_i

We removed $k = n_i - n_{i-1}$ smallest items $\leq \frac{3k}{n_i}$

Total Size of items in G_i ≤ 3

- Consider k smallest elements in G_i.
- Suppose their total size $> \frac{3k}{n_i}$

$\Rightarrow \underbrace{\exists \text{ an item with size}} > \frac{3}{n_i}$

$\Rightarrow$ Total size of $n_i - k$ largest items in G_i $\Big\} > 3\left(\frac{n_i - k}{n_i}\right)$

$\Rightarrow$ Total size in G_i $> 3\left(\frac{n_i - k}{n_i}\right) + \frac{3k}{n_i} = 3$

Total size of $G_i \leq 3$ $\quad \boxed{n_r \leq 3 \cdot \text{SIZE}(I)}$

$\therefore$ Size of removed items from G_i $\Big\} \leq 3\left(\dfrac{n_i - n_{i-1}}{n_i}\right)$

$\dfrac{1}{j} \geq \dfrac{1}{n_i} \quad \forall j \leq n_i$ $\qquad \leq 3 \cdot \displaystyle\sum_{j=n_{i-1}+1}^{n_i} \frac{1}{j}$

$\therefore$ Total size of all items removed $\Big\} \leq 3 \underbrace{\displaystyle\sum_{j=1}^{n_r} \frac{1}{j}}_{H_{n_r}} + b$

$= O(\log(\text{SIZE}(I)))$

Algorithm

for instance I^*, first fit uses $\leq 2 \cdot SIZE(I^*) + 1$ bins

BINPACK (I)

if SIZE(I) < 10 :

Pack pieces using First Fit

$\leq 20\,PT + 1$ guarantee

else

- Apply grouping to create I'
- Pack discards using First Fit in $O(\log SIZE(I))$ bins

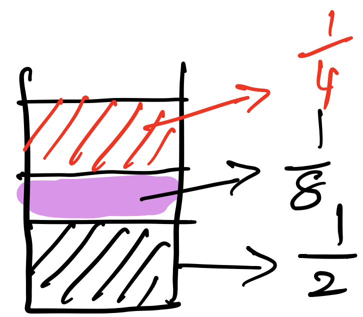
$\longrightarrow$ Why is this possible?

- Solve LP on I'
- Let x be optimal soln.
- Pack $\lfloor x_j \rfloor$ bins in configuration T_j for $j = 1, 2, \dots, N$; call the packed pieces I_1

integer part of solution

- Pack the remaining pieces, I_2, via BINPACK (I_2).

2.3 bins packed like



Lemma

$$Z^*_{LP}(I_1) + Z^*_{LP}(I_2) \overset{(2)}{\leq} Z^*_{LP}(I') \overset{(1)}{\leq} Z^*_{LP}(I)$$

Proof:

(1) is straightforward.

(2) Optimal soln. for I' is x

$\rightarrow$ $x - \lfloor x \rfloor = (\dots, x - \lfloor x_j \rfloor, \dots)$ is a feasible soln. for LP on I_2

$\rightarrow$ $\lfloor x \rfloor$ is a feasible solution for I_1.

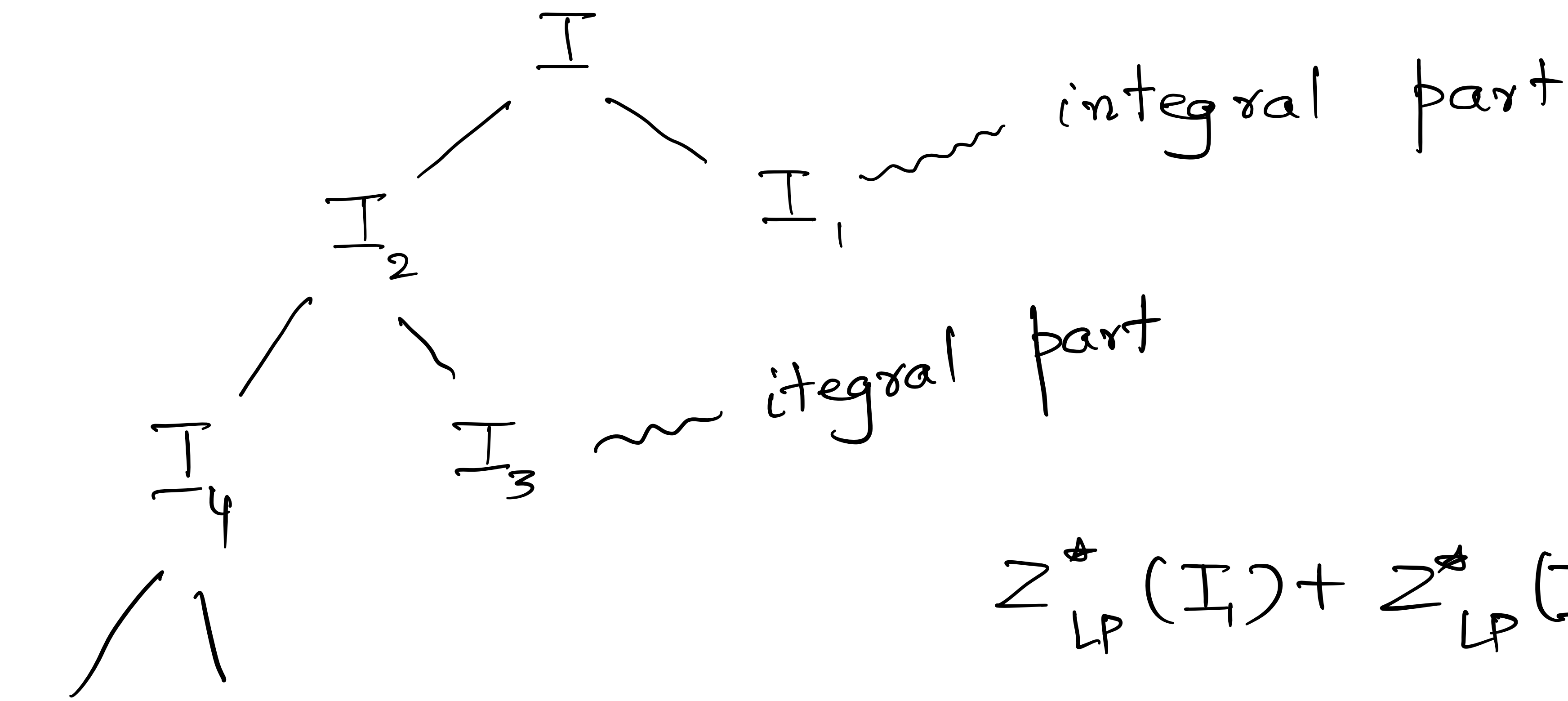
- Each recursion level

first part ← { * pieces packed by integer part of LP solution

{ * pieces packed after being discarded by grouping → need to account for

* pieces packed by recursive call.

- Total #bins occupied by pieces in first part over all recursion levels } $\leq Z^*_{LP}(I)$



I_ℓ . . . $\text{SIZE}(I_\ell) \le 10$

$$Z^*_{LP}(I_1) + Z^*_{LP}(I_2) \le Z^*_{LP}(I)$$

$$Z^*_{LP}(I_2) \ge Z^*_{LP}(I_3) + Z^*_{LP}(I_4)$$

$$Z^*_{LP}(I) \ge Z^*_{LP}(I_1) + Z^*_{LP}(I_3) + Z^*_{LP}(I_5) + \cdots +$$

- Only error in each recursion level caused by discarded pieces

- Need to bound # recursion levels

$SIZE(I_2) \leq \frac{1}{2} SIZE(I)$

- Total size of input in recursive call is $\leq \frac{1}{2} \cdot$ Size of original input

- $SIZE(I_2) \leq \sum_{j=1}^{N} x_j - \lfloor x_j \rfloor$

- RHS above $\leq$ # non-zeroes in the optimal LP soln. x

basic optimal solution

(Reading exercise)

$\leq$ # constraints in LP

$\leq$ # distinct item sizes after grouping

$$\leq \frac{SIZE(I)}{2}$$

$\boxtimes$

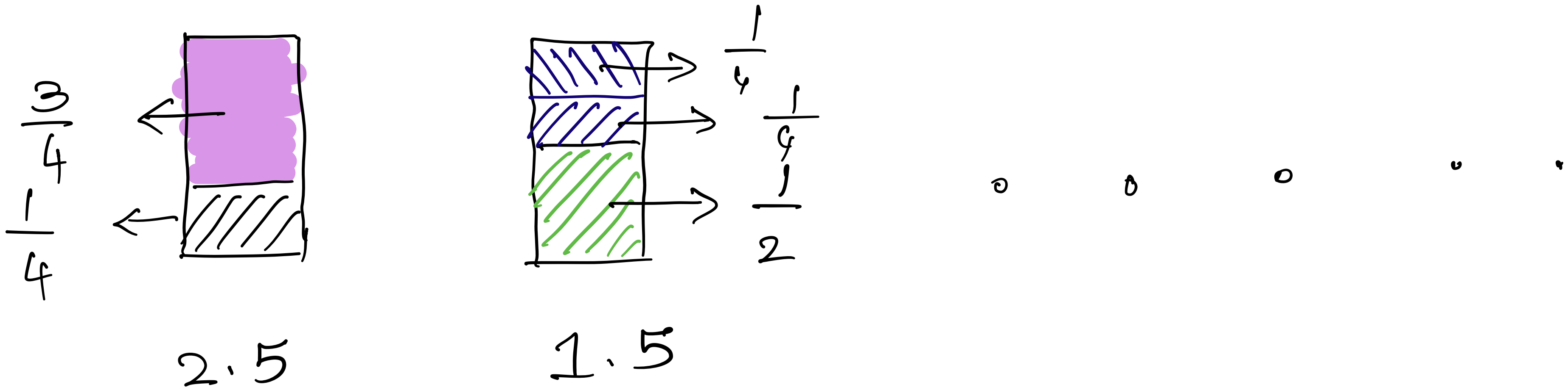
$$\left.\begin{array}{l}\text{Total \#bins used by} \\ \quad\text{algo}\end{array}\right\}$$

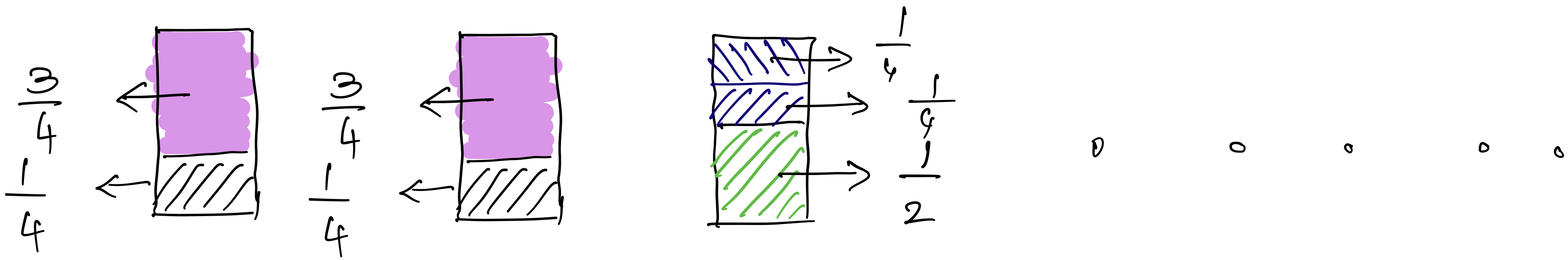
$$\leq \begin{array}{l}\text{Total \#bins used to} \\ \text{pack the integral} \\ \quad\text{part}\end{array} + \begin{array}{l}\text{Total \#bins used} \\ \text{to pack the} \\ \text{discarded pieces}\end{array}$$

$$\leq Z^{*}_{LP}(I) + O(\log^2 SIZE(I))$$

$$\leq OPT(I) + O(\log^2 OPT(I))$$

$\boxtimes$





2 items of size $\frac{3}{4}$, 1 item of size $\frac{1}{2}$, 4 items of size $1/4$